Different paths to some fundamental physical laws: relativistic polarization of a moving magnetic dipole

2010 Eur. J. Phys. 31 1233

(http://iopscience.iop.org/0143-0807/31/5/022)

IOP PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **31** (2010) 1233–1238 doi:10.1088/0143-0807/31/5/022

Different paths to some fundamental physical laws: relativistic polarization of a moving magnetic dipole

Alexander L Kholmetskii[1,2] **and T Yarman**[2]

[1] Belarusian State University, Minsk, Belarus
[2] Okan University, Akfirat, Istanbul, Turkey

E-mail: tyarman@gmail.com

Received 27 June 2010, in final form 23 July 2010
Published 31 August 2010
Online at stacks.iop.org/EJP/31/1233

Abstract

In this paper we consider the relativistic polarization of a moving magnetic dipole and show that this effect can be understood via the relativistic generalization of Kirchhoff's first law to a moving closed circuit with a steady current. This approach allows us to better understand the law of relativistic transformation of four-current density when it is applied to the moving macroscopic magnetic dipoles.

1. Introduction

It is known that many laws of fundamental physics may admit various paths to their derivation, and the awareness of all such paths is helpful for better understanding the physical meaning of these laws. In addition, a finding of any novel way to prove the validity of any law may enrich its comprehension and thus its perceived physical content, and it is certainly very useful from an educational viewpoint, both for undergraduate and graduate students. The present paper is directed in the first instance to these students.

In the following, we propose to consider the law of relativistic polarization of a moving magnetic dipole, focusing our attention on the well-known relationship of this effect with the relativistic transformation of four-current density in a circuit carrying a steady current. We show that the popular approach suggested by Feynman [1] consisting in associating the transformation in question with the scale contraction effect in a moving straight wire is, in general, insufficient for a full understanding of the relativistic polarization of moving magnetic dipoles. In particular, we additionally involve Kirchhoff's first law generalized to a moving closed circuit and show that the law of conservation of total charge for such a circuit leads to its polarization in an inertial frame, wherein the circuit is moving with the non-vanished constant velocity (section 2). Finally, in section 3 we further discuss the physical meaning of the results obtained.

2. On the dipole moment of a moving loop carrying a steady current

As known, the motion of a magnetic dipole with velocity $\boldsymbol{v}$ and magnetic moment $\boldsymbol{\mu}$ induces the appearance of an electric dipole moment

$$\boldsymbol{p} = \frac{\boldsymbol{v} \times \boldsymbol{\mu}}{c^2} \tag{1}$$

in the inertial frame of observer K, where c is the light velocity in vacuum. Equation (1) can be characterized as the relativistic polarization of a magnetic dipole, and it follows from the transformation of the polarization–magnetization tensor (see e.g. [2]). The physical meaning of equation (1) is discussed in a number of books and papers from different viewpoints (e.g. [2–4]).

Formally, equation (1) can be obtained from the Lorentz transformation of the charge density ρ and current density $\boldsymbol{j}$, constituting a four-vector, which in the one-dimensional case takes the form

$$j_x = \gamma(j_{0x} + \rho_0 v), \tag{2}$$

$$\rho = \gamma\left(\rho_0 + \frac{j_{0x} v}{c^2}\right), \tag{3}$$

where j_{0x}, ρ_0 are measured in the inertial frame K_0, moving at constant velocity v along the x-axis of a laboratory frame K, and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

Indeed, let us apply equation (3) to a neutral conductive rectangular loop with a steady current I, lying in the x-y plane and moving at the constant velocity $\boldsymbol{v}\{v, 0, 0\}$ (see figure 1). We adopt the following notation:

ρ_0 is the total charge density of segments of the loop in its rest frame K_0;
j_0 is the current density in the loop for an observer in the frame K_0;
I is the current in the loop for an observer in K_0;
ρ_{0e} is the charge density of conduction electrons in the wire of the loop in K_0;
ρ_{0+} is the charge density of positive ions in the wire of the loop in K_0;
l_0 is the length of the segment of the loop in K_0;
S_0 is the area of the cross section of the conducting wire of the loop in K_0;
$\boldsymbol{u}_0$ is the flow velocity of conduction electrons in K_0;
ρ_{down}, ρ_{up}, ρ_{left}, ρ_{right} are the total charge densities in the down, upper, left and right segments of the loop, respectively, as viewed by the laboratory observer K;
ρ_+ is the charge density of positive ions in the wire of the loop in the laboratory frame K;
$(\rho_{\text{down}})_e$, $(\rho_{\text{up}})_e$, $(\rho_{\text{left}})_e$, $(\rho_{\text{right}})_e$ are the charge densities of conduction electrons in corresponding segments of the loop in the frame K;
S_{down}, S_{up}, S_{left}, S_{right} are the areas of the cross section of the conducting wire for corresponding segments of the loop in the frame K;
Q_{up}, Q_{down} are the charges of the upper and down segments of the loop, respectively, in the frame K;
$\boldsymbol{u}_{\text{down}}$, $\boldsymbol{u}_{\text{up}}$, $\boldsymbol{u}_{\text{left}}$, $\boldsymbol{u}_{\text{right}}$ are the flow velocities of conduction electrons in the corresponding segments of the loop in the frame K;
$\boldsymbol{p}$ is the electric dipole moment of the loop in the frame K.

Thus in the adopted designation, the proper magnetic moment of the loop takes the form $\mu_{0z} = I l_0^2$.

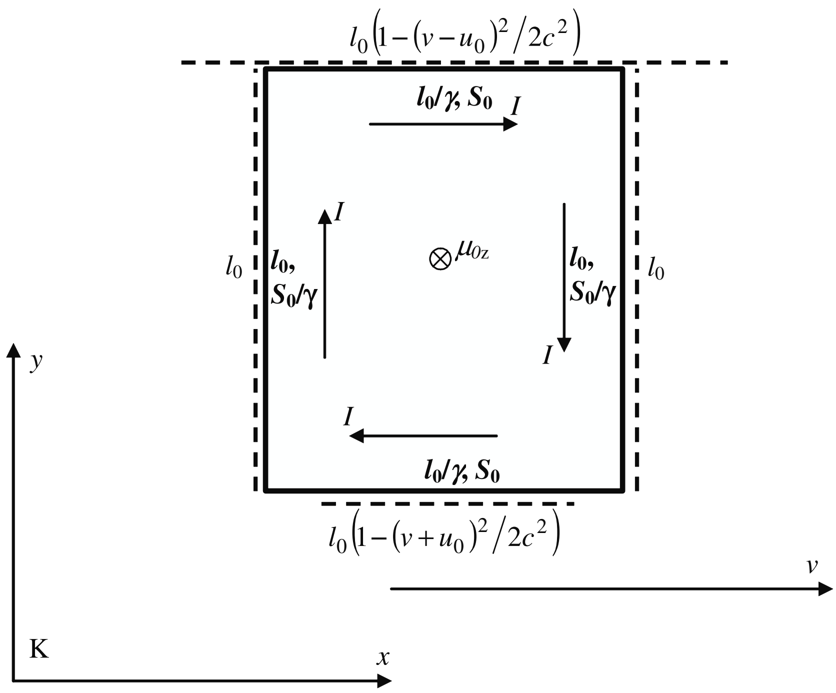

Figure 1. Rectangular conducting loop lying in the x-y plane moves at constant velocity v along the x-axis of the laboratory frame K. A proper length of each segment of the loop is equal to l_0, the proper area of the cross section of the conducting wire is S_0. The proper magnetic moment of the loop is equal to $\mu_0 = Il_0^2$ and lies in the negative z-direction, where I is the current to be measured in the rest frame of the loop. The length and cross-sectional area of each section of the moving loop are shown in bold. We also show contraction of the filaments of conduction electrons (dotted lines) in various sections of the moving rectangular loop according to [5].

Since in the rest frame of the loop $\rho_0 = 0$, equation (3) yields

$$\rho_{\text{down}} = -\frac{\gamma j_0 v}{c^2}, \tag{4}$$

$$\rho_{\text{up}} = \frac{\gamma j_0 v}{c^2}, \tag{5}$$

$$\rho_{\text{left}} = \rho_{\text{right}} = 0, \tag{6}$$

where $j_0 = I/S_0$. Hence the electric dipole moment of the moving loop becomes

$$p_y = \frac{Q_{\text{up}} - Q_{\text{down}}}{2} l_0, \tag{7}$$

where Q_{up} (Q_{down}) is the electric change of the upper (down) segment of the moving loop. Insofar as $Q_{\text{up}} = \rho_{\text{up}} S_0 \frac{l_0}{\gamma}$, $Q_{\text{down}} = \rho_{\text{down}} S_0 \frac{l_0}{\gamma}$, based on (4) and (5), we obtain

$$p_y = \frac{Q_{\text{up}} - Q_{\text{down}}}{2} l_0 = \frac{\gamma j_0 v}{c^2} \frac{S_0 l_0}{\gamma} l_0 = \frac{v \mu_{0z}}{c^2}, \tag{8}$$

which is the particular case of equation (1) for the orthogonal $\boldsymbol{\mu}$ and $\boldsymbol{v}$.

This derivation of equation (1) is present in many textbooks, but it does not clarify the physical meaning of equations (2) and (3). Panofsky and Phillips [2], as well as Rosser [3] explain these equations through the relativity of simultaneity of events in different inertial frames. This is a correct but rather formal explanation. Feynman in his lectures [1] demonstrates the validity of transformations (2) and (3) for the particular case where a very long straight wire with a steady current is moving along its axis at the constant velocity v. He involves the relativistic effect of length contraction, paying attention to the fact that for a laboratory observer the velocities of positive ions (composing the lattice of wire) and negative conduction electrons differ from each other due to the flow of these electrons along the wire. Hence, the length contraction effect is also different for the negative and positive charges and as a result, the wire acquires the non-zero net charge according to equation (3). At the same time, Feynman did not mention that his analysis is relevant for a model problem, dealing with the artificial infinitely long wire. Maybe this circumstance explains why some authors continue to apply the same line of reasoning to moving finite-size closed circuits carrying a steady current.

In a consistent way, the idea of explaining the transformations (2) and (3) via the length contraction effect in a current loop has been analysed in [5]. The author of [5] observed that the velocity of the conduction electrons, constituting the current I, exceeds v in the down section of the circuit, but is less than v in the upper section. (From now on, we take into account that the direction of the flow velocity of conduction electrons is opposite to the direction of the current shown in figure 1.) Hence, he obtained that the length of the filament of current in the down section is $l_0\left(1-\frac{(v+u_0)^2}{2c^2}\right)$, while in the upper section, it is $l_0\left(1-\frac{(v-u_0)^2}{2c^2}\right)$ to an accuracy of the order c^{-2}, where u_0 is the flow velocity of conduction electrons in the rest frame of the loop. Correspondingly, the charge density of conduction electrons in the down section is

$$(\rho_{\text{down}})_e=\frac{\rho_{0e}}{1-\frac{(v+u_0)^2}{2c^2}},$$

whereas in the upper section it becomes

$$(\rho_{\text{up}})_e=\frac{\rho_{0e}}{1-\frac{(v-u_0)^2}{2c^2}},$$

where we have introduced the charge density of conduction electrons ρ_{0e} in the rest frame of the loop. In this way, Blackford also arrived at equations (2) and (3) and finally at equation (1), at least within the accuracy of the calculations c^{-2}. However, in our opinion, this way of reasoning is unphysical. One can see that within such an approach, the corners of the circuit represent the points of discontinuity of current (figure 1); besides, the length of the filament of current in the upper section becomes inadmissibly longer than the length of this section itself.

At the same time, there is a simple and, in our opinion, physically correct way to derive equation (1), which is based on Einstein's law of velocity composition and Kirchhoff's first law extended to relativistically moving circuits. First, we determine the velocities of conduction electrons along each segment of the loop, as seen by a laboratory observer, applying Einstein's law of velocity composition:

$$(u_{\text{down}})_x=\frac{v+u_0}{1+u_0v/c^2}, \tag{9}$$

$$(u_{\text{up}})_x=\frac{v-u_0}{1-u_0v/c^2}, \tag{10}$$

$$(u_{\text{left}})_y=-\frac{u_0}{\gamma}, \tag{11}$$

$$(u_{\mathrm{right}})_y = \frac{u_0}{\gamma}. \tag{12}$$

Now consider a flow of current, for example, near the lower-right corner of the loop. Since no charge is accumulated on this corner, we conclude that during a unit time interval, a number of conduction electrons entering into this corner from the down side of the loop $((\rho_{\mathrm{down}})_e((u_{\mathrm{down}})_x - v)S_{\mathrm{down}})$ is equal to the number of electrons leaving this corner and entering into the right side $((\rho_{\mathrm{right}})_e u_{\mathrm{right}} S_{\mathrm{right}})$. Equating these quantities, we obtain a relativistic generalization of Kirchhoff's first law:

$$(\rho_{\mathrm{down}})_e((u_{\mathrm{down}})_x - v)S_{\mathrm{down}} = (\rho_{\mathrm{right}})_e(u_{\mathrm{right}})_y S_{\mathrm{right}}, \tag{13}$$

which in its physical meaning represents the continuity equation for the conduction electrons.

Further on, we take into account that due to the scale contraction effect, $S_{\mathrm{down}}/S_{\mathrm{right}} = \gamma$, and insert the expressions for $(u_{\mathrm{down}})_x$ (equation (9)) and $(u_{\mathrm{right}})_y$ (equation (12)) into equation (13). Then, one obtains

$$(\rho_{\mathrm{down}})_e = (\rho_{\mathrm{right}})_e(1 + u_0 v/c^2). \tag{14}$$

Applying Kirchhoff's first law to the upper-right corner of loop, we derive in a similar way

$$(\rho_{\mathrm{up}})_e = (\rho_{\mathrm{right}})_e(1 - u_0 v/c^2). \tag{15}$$

Analogously, for the upper-left corner we obtain

$$(\rho_{\mathrm{up}})_e = (\rho_{\mathrm{left}})_e(1 - u_0 v/c^2). \tag{16}$$

Now let us compute the total charge Q of conduction electrons in the loop, which is the invariant quantity

$$(\rho_{\mathrm{down}})_e S_0 \frac{l_0}{\gamma} + (\rho_{\mathrm{right}})_e \frac{S_0}{\gamma} l_0 + (\rho_{\mathrm{up}})_e S_0 \frac{l_0}{\gamma} + (\rho_{\mathrm{left}})_e \frac{S_0}{\gamma} l_0 = Q = 4\rho_{0e} S_0 l_0. \tag{17}$$

(We have taken into account that in the rest frame of the loop $Q = 4\rho_{0e} S_0 l_0$.) Combining equations (14)–(17), we obtain

$$(\rho_{\mathrm{right}})_e = (\rho_{\mathrm{left}})_e = \gamma \rho_{0e}, \tag{18}$$

which agrees with the transformation rule (3) for $j_x = 0$.

The charge density for positive ions has the value

$$\rho_+ = \gamma \rho_{0+} \tag{19}$$

in all the segments of the moving loop, where ρ_{0+} is measured in its rest frame K_0.

The equations (14)–(19) allow us to determine the total charge density ρ in each segment of the loop, if we take into account that $\rho_{0\mathrm{e}} + \rho_{0+} = 0$:

$$\rho_{\mathrm{down}} = -\gamma \rho_{0e} \frac{u_0 v}{c^2}, \qquad \rho_{\mathrm{up}} = \gamma \rho_{0e} \frac{u_0 v}{c^2}, \qquad \rho_{\mathrm{left}} = \rho_{\mathrm{right}} = 0.$$

The total charges of the down and upper sections are

$$Q_{\mathrm{down}} = \rho_{\mathrm{down}} S_0 \frac{l_0}{\gamma} = -\frac{I l_0}{c^2}, \qquad Q_{\mathrm{up}} = \rho_{\mathrm{up}} S_0 \frac{l_0}{\gamma} = \frac{I l_0}{c^2}.$$

Inserting these values into equation (7), we arrive at equation (1).

In a similar way, one can show the validity of equation (1) for the case where the vectors $\boldsymbol{v}$ and $\boldsymbol{\mu}$ are not collinear with each other.

The presented derivation of equation (1) can be directly extended to a plain circuit of arbitrary shape, because a flat area enclosed by the circuit can always be divided into a large number of sufficiently small rectangular circuits, where the currents mutually cancel each other for neighbouring sections. Thus, a non-compensated current is present only on the boundary of the area, that is, in the plain circuit. Hence, consistently applying equation (1) to each of the small rectangular elements, we arrive at equation (1) for the entire circuit in question.

3. Conclusion

We have shown that the electric polarization of a moving classical magnetic dipole can be understood on the basis of the relativistic generalization of Kirchhoff's first law to a closed circuit via involving Einstein's law of speed composition. Due to the latter law, for a laboratory observer, the difference of the velocities (Δv) of a closed circuit as a whole, and conduction electrons flowing in the circuit do not coincide, in general, with the flow velocity of these electrons in the proper (rest) frame of the circuit. In particular, for the case shown in figure 1, the modulus of Δv is larger in the top section of the circuit than in its bottom section. Therefore, due to the continuity equation for the conduction electrons (being expressed via Kirchhoff's first law), the charge density of conduction electrons *must be* larger in the bottom section in comparison with the top section, as viewed by a laboratory observer. Hence we get the electric polarization of the moving circuit in a laboratory reference frame.

At the fundamental level, this effect is rooted, of course, to the relativity of simultaneity of events in different inertial frames, as mentioned in the books by Panofsky and Phillips [2], Rosser [3], as well as in other textbooks.

At the same time, our classical derivation of equation (1) cannot be directly applied to the particles with spin (electron, neutron, etc): in this case we just have to adopt, as a universal relativistic result, that a moving particle with the magnetic moment $\boldsymbol{\mu}$ has the electric dipole moment $\boldsymbol{p}$ in the frame of observation.

References

[1] Feynman R P, Leighton R B and Sands M 1964 *The Feynman Lectures on Physics* vol 2 (Reading, MA: Addison-Wesley)

[2] Panofsky W K H and Phillips M 1962 *Classical Electricity and Magnetism* 2nd edn (New York: Wiley)

[3] Rosser W G V 1964 *An Introduction to the Theory of Relativity* (London: Butterworths)

[4] Fisher G P 1971 *Am. J. Phys.* **39** 1528

[5] Blackford B L 1994 *Am. J. Phys.* **62** 1005